Polynomial notes

(1) Polynomial : The expression which contains one or more terms with non-zero coefficient is called a polynomial. A polynomial can have any number of terms.
***For Example*:** 10, $a + b$, $7x + y + 5$, $w + x + y + z$, etc. are some polynomials.

(2) Degree of polynomial : The highest power of the variable in a polynomial is called as the degree of the polynomial.
***For Example*:** The degree of $p(x) = x^5 - x^3 + 7$ is 5.

(3) Linear polynomial : A polynomial of degree one is called a linear polynomial.
***For Example*:** $1/(2x - 7)$, $\sqrt{s} + 5$, etc. are some linear polynomial.

(4) Quadratic polynomial : A polynomial having highest degree of two is called a quadratic polynomial. The term 'quadratic' is derived from word 'quadrate' which means square. In general, a quadratic polynomial can be expressed in the form $ax^2 + bx + c$, where $a \neq 0$ and a, b, c are constants.
***For Example*:** $x^2 - 9$, $a^2 + a + 7$, etc. are some quadratic polynomials.

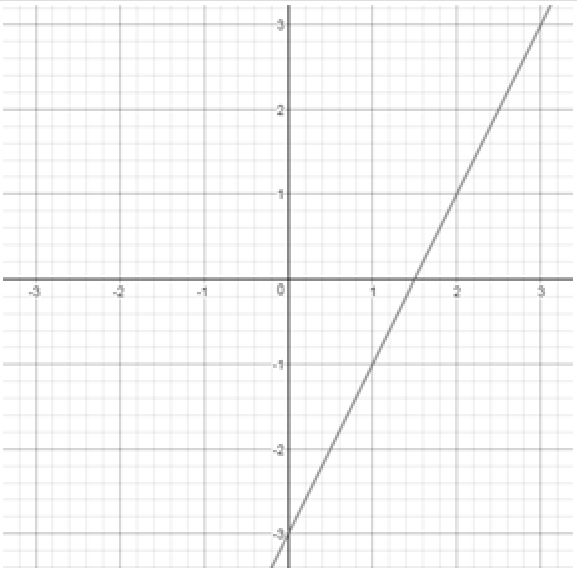
(5) Cubic Polynomial : A polynomial having highest degree of three is called a cubic polynomial. In general, a quadratic polynomial can be expressed in the form $ax^3 + bx^2 + cx + d$, where $a \neq 0$ and a, b, c, d are constants.
***For Example*:** $x^3 - 9x + 2$, $a^3 + a^2 + \sqrt{a} + 7$, etc. are some cubic polynomial.

(6) Zeroes of a Polynomial : The value of variable for which the polynomial becomes zero is called as the zeroes of the polynomial. In general, if k is a zero of $p(x) = ax + b$, then $p(k) = ak + b = 0$, i.e., $k = -b/a$. Hence, the zero of the linear polynomial $ax + b$ is $-b/a = -$(Constant term)/(coefficient of x)
***For Example*:** Consider $p(x) = x + 2$. Find zeroes of this polynomial.
If we put $x = -2$ in $p(x)$, we get,
$p(-2) = -2 + 2 = 0$.
Thus, -2 is a zero of the polynomial $p(x)$.

(7) Geometrical Meaning of the Zeroes of a Polynomial:
(i) For Linear Polynomial:
In general, for a linear polynomial $ax + b$, $a \neq 0$, the graph of $y = ax + b$ is a straight line which intersects the x-axis at exactly one point, namely, $(-b/a , 0)$. Therefore, the linear polynomial $ax + b$, $a \neq 0$, has exactly one zero, namely, the x-coordinate of the point where the graph of $y = ax + b$ intersects the x-axis.

For Example**:** The graph of $y = 2x - 3$ is a straight line passing through points (0, -3) and (3/2, 0).

x	0	3/2
$y = 2x - 3$	6	0

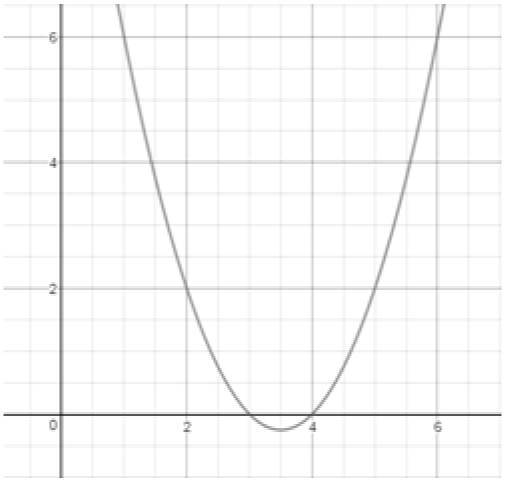


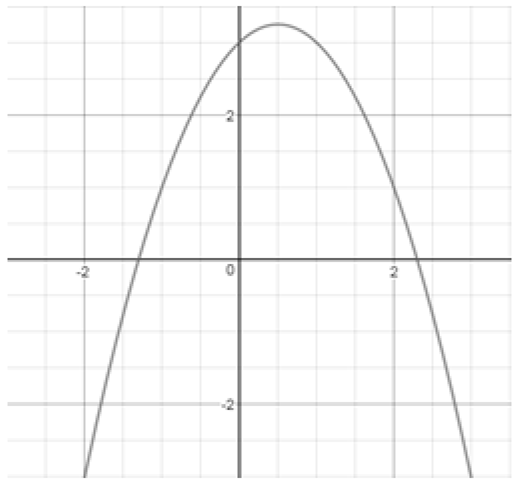
Here, the graph of $y = 2x - 3$ is a straight line which intersects the x-axis at exactly one point, namely, (3/2 , 0).

(ii) For Quadratic Polynomial:

In general, for any quadratic polynomial $ax^2 + bx + c$, $a \neq 0$, the graph of the corresponding equation $y = ax^2 + bx + c$ has one of the two shapes either open upwards like curve or open downwards like curve depending on whether $a > 0$ or $a < 0$. (These curves are called parabolas.)

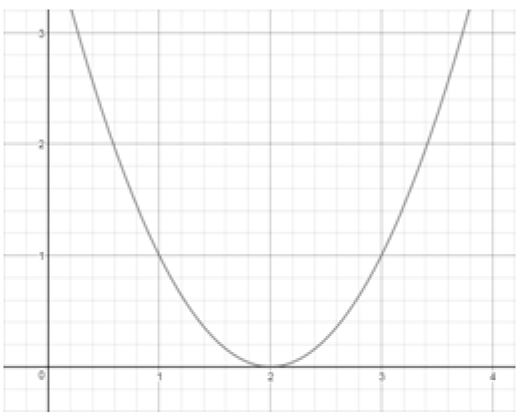
Case 1**:** The Graph cuts x-axis at two distinct points.

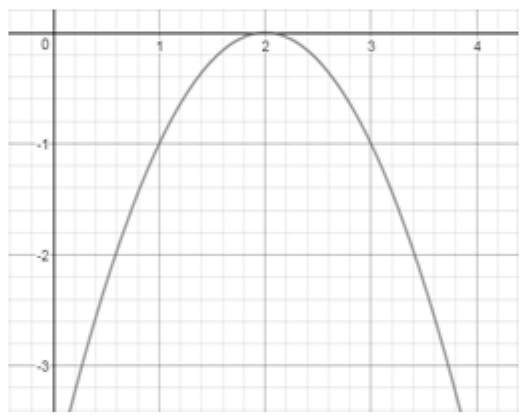




The x-coordinates of the quadratic polynomial $ax^2 + bx + c$ have two zeros in this case.

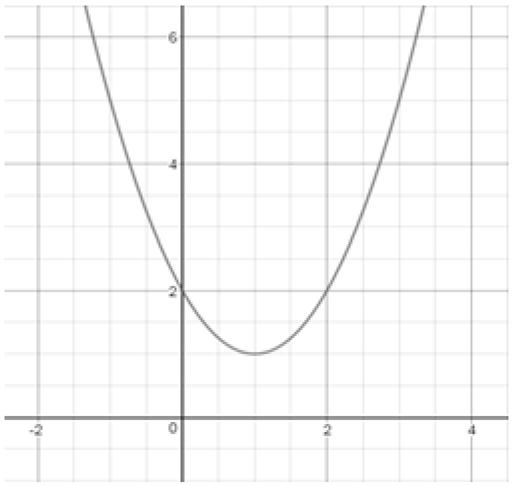
Case 2**:** The Graph cuts x-axis at exactly one point.

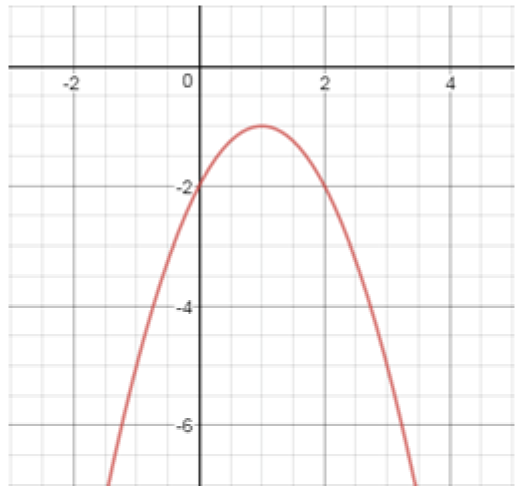




The x-coordinates of the quadratic polynomial $ax^2 + bx + c$ have only one zero in this case.

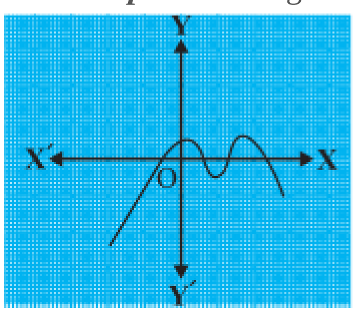
Case 3**:** The Graph is completely above x-axis or below x-axis.





The quadratic polynomial $ax^2 + bx + c$ have no zero in this case.

For Example**:** For the given graph, find the number of zeroes of p(x).



From the figure, we can see that the graph intersects the x-axis at four points.
Therefore, the number of zeroes is 4.

(8) Relationship between Zeroes and Coefficients of a Polynomial:

(i) Quadratic Polynomial:

In general, if α and β are the zeroes of the quadratic polynomial $p(x) = ax^2 + bx + c$, $a \neq 0$, then we know that $(x - \alpha)$ and $(x - \beta)$ are the factors of p(x).
Moreover, $\alpha + \beta = -b/a$ and $\alpha\,\beta = c/a$.
In general, sum of zeros = -(Coefficient of x)/(Coefficient of x^2).
Product of zeros = (Constant term)/ (Coefficient of x^2).

For Example**:** Find the zeroes of the quadratic polynomial $x^2 + 7x + 10$, and verify the relationship between the zeroes and the coefficients.
On finding the factors of $x^2 + 7x + 10$, we get, $x^2 + 7x + 10 = (x + 2)(x + 5)$
Thus, value of $x^2 + 7x + 10$ is zero for $(x+2) = 0$ or $(x+5) = 0$. Or in other words, for $x = -2$ or $x = -5$.
Hence, zeros of $x^2 + 7x + 10$ are -2 and -5.

Now, sum of zeros = -2 + (-5) = -7 = -7/1 = -(Coefficient of x)/(Coefficient of x^2). Similarly, product of zeros = (-2) x (-5) = 10 = 10/1 = (Constant term)/ (Coefficient of x^2).

***For Example*:** Find the zeroes of the quadratic polynomial t^2 -15, and verify the relationship between the zeroes and the coefficients.
On finding the factors of t^2 -15, we get, t^2 -15= (t + √15) (t - √15)
Thus, value of t^2 -15 is zero for (t +√15) = 0 or (t - √15) = 0. Or in other words, for t = √15 or t = -√15.
Hence, zeros of t^2 -15 are √15 and -√15.
Now, sum of zeros = √15 + (-√15) = 0 = -0/1 = -(Coefficient of t)/(Coefficient of t^2). Similarly, product of zeros = (√15) x (-√15) = -15 = -15/1 = (Constant term)/ (Coefficient of t^2).

***For Example*:** Find a quadratic polynomial for the given numbers as the sum and product of its zeroes respectively 4, 1.
Let the quadratic polynomial be $ax^2 + bx + c$.
Given, α + β = 4 = 4/1 = -b/a.
α β = 1 = 1/1 = c/a.
Thus, a = 1, b = -4 and c = 1.
Therefore, the quadratic polynomial is $x^2 - 4x + 1$.

(ii) Cubic Polynomial: In general, it can be proved that if α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, then,
α + β + γ = –b/a ,
αβ + βγ + γα = c/a and α β γ = – d/a .

(9) Division Algorithm for Polynomials : If p(x) and g(x) are any two polynomials with g(x) ≠ 0, then we can find polynomials q(x) and r(x) such that p(x) = g(x) × q(x) + r(x), where r(x) = 0 or degree of r(x) < degree of g(x).

***For Example*:** Divide $3x^2 – x^3 – 3x + 5$ by $x – 1 – x^2$, and verify the division algorithm.
On dividing $3x^2 – x^3 – 3x + 5$ by $x – 1 – x^2$, we get,

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$$\begin{array}{r|l} & x - 2 \\ \hline -x^2 + x - 1 & -x^3 + 3x^2 - 3x + 5 \\ & -x^3 + x^2 - x \\ & + \quad - \quad + \\ \hline & \quad 2x^2 - 2x + 5 \\ & \quad 2x^2 - 2x + 2 \\ & \quad - \quad + \quad - \\ \hline & \qquad\qquad 3 \end{array}$$

Here, quotient is (x – 2) and remainder is 3.
Now, as per the division algorithm, Divisor x Quotient + Remainder = Dividend

LHS = $(-x^2 + x + 1)(x - 2) + 3$
= $(-x^3 + x^2 - x + 2x^2 - 2x + 2 + 3)$
= $(-x^3 + 3x^2 - 3x + 5)$
RHS = $(-x^3 + 3x^2 - 3x + 5)$
Thus, division algorithm is verified.

For Example**:** On dividing $x^3 - 3x^2 + x + 2$ by a polynomial g(x), the quotient and remainder were $(x - 2)$ and $(-2x + 4)$, respectively. Find g(x).
Given, dividend = p(x) = $(x^3 - 3x^2 + x + 2)$, quotient = $(x - 2)$, remainder = $(-2x + 4)$.
Let divisor be denoted by g(x).
Now, as per the division algorithm,
Divisor x Quotient + Remainder = Dividend
$(x^3 - 3x^2 + x + 2) = g(x)\,(x - 2) + (-2x + 4)$
$(x^3 - 3x^2 + x + 2 + 2x - 4) = g(x)\,(x - 2)$
$(x^3 - 3x^2 + 3x - 2) = g(x)\,(x - 2)$
Hence, g(x) is the quotient when we divide $(x^3 - 3x^2 + 3x - 2)$ by $(x - 2)$.

$$
\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
x-2 & x^3 - 3x^2 + 3x - 2 \\
 & x^3 - 2x^2 \\
 & - \quad + \\
\hline
 & \quad -x^2 + 3x - 2 \\
 & \quad -x^2 + 2x \\
 & \quad + \quad - \\
\hline
 & \qquad\qquad x - 2 \\
 & \qquad\qquad x - 2 \\
 & \qquad\qquad - \quad + \\
\hline
 & \qquad\qquad 0 \\
\hline
\end{array}
$$

Therefore, g(x) = $(x^2 - x + 1$